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ORIGINAL RESEARCH

Positive solutions to singular semipositone boundary value problems of second order coupled differential systems

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Abstract We establish the existence of positive solutions for the second order singular semipositone coupled Dirichlet systems

$$\begin{cases} x'' + f_1\big(t, y(t)\big) + e_1(t) = 0, \\ y'' + f_2\big(t, x(t)\big) + e_2(t) = 0, \\ x(0) = x(1) = 0, \qquad y(0) = y(1) = 0. \end{cases}$$

The proof relies on Schauder's fixed point theorem.

Keywords Positive solutions · Semipositone boundary value problem · Singular coupled Dirichlet systems · Schauder's fixed point theorem

Mathematics Subject Classification 34B15 · 34B16 · 34L30

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1 Introduction

In this paper, we study the existence of positive solutions for the second order singular semipositone coupled differential systems

$$\begin{cases} x'' + f_1\big(t, y(t)\big) + e_1(t) = 0, \\ y'' + f_2\big(t, x(t)\big) + e_2(t) = 0, \\ x(0) = x(1) = 0, \qquad y(0) = y(1) = 0. \end{cases} \tag{1.1}$$

Here $e_1, e_2 \in C[0,1]$, $f_1, f_2 \in C([0,1] \times (0,+\infty), (0,+\infty))$ and may be singular near the zero. A solution of (1.1) is a pair $(x(t), y(t))$ of continuous functions on [0,1], twice differentiable on (0,1), with $x(t), y(t) > 0$ for $0 < t < 1$ and $x(0) = x(1) = y(0) = y(1) = 0$ such that $x''(t) + f_1(t, y(t)) + e_1(t) = 0$ and $y''(t) + f_2(t, x(t)) + e_2(t) = 0$ for all $t \in (0,1)$.

There has been increasing interest in the subject of singular differential equations due to its strong application background, and consequently, a number of theoretical results for the solution of various types of singular differential equations have been developed, we refer the reader to [1–11] and the references therein. Boundary value problems of singular differential systems have been studied extensively by many authors over the last two decades [2–11]. In [4–10], the authors establish the conditions for the existence of positive solutions of a singular boundary value problem with second-order differential systems.

In [12, 13], Cao, Jiang et al. establish the existence of periodic solutions for the second order non-autonomous singular coupled systems

$$\begin{cases} x'' + a_1(t)x = f_1\big(t, y(t)\big) + e_1(t), \\ y'' + a_2(t)y = f_2\big(t, x(t)\big) + e_2(t). \end{cases}$$

The proof relies on Schauder's fixed point theorem. Some recent results in the literature are generalized and improved.

In [10], Zhang et al. consider with a nonlinear singular coupled differential system with four-point boundary conditions

$$\begin{cases} -x'' = f(t, y(t)), \\ -y'' = g(t, x(t)), \\ \alpha x(0) - \beta x'(0) = \delta x(1) + \gamma x'(1) = 0, \\ y(0) = ay(\xi_1), \qquad y(1) = by(\xi_2), \end{cases}$$

where $0 < \xi_1 < \xi_2 < 1$; $\alpha, \beta, \gamma, \delta, a, b$ are nonnegative constants such that $\rho = \beta\gamma + \alpha\gamma + \alpha\delta > 0$. By using Schauder fixed point theorem, and established a necessary and sufficient condition for the existence of positive solutions.

In [11], by employing a fixed point index theorem, Zhang et al. study the existence of positive solutions for a singular semipositone coupled differential system

$$\begin{cases} -x'' = f(t, y(t)) + q(t), \\ -y'' = g(t, x(t)), \\ x(0) = x(1) = 0, \\ \alpha y(0) - \beta y'(0) = \delta y(1) + \gamma y'(1) = 0, \end{cases}$$

where $\alpha, \beta, \gamma, \delta$ are nonnegative constants such that $\rho = \beta\gamma + \alpha\gamma + \alpha\delta > 0$. A new existence of positive solutions result is established.

Motivated by the work of the above papers, the aim of this paper is to establish some simple criteria for the existence of positive solution to Dirichlet BVP of the singular semipositone coupled differential systems (1.1).

The remaining part of the article is organized as follows. In section "preliminaries", some preliminary results will be given. In the remaining sections, by employing a basic application of Schauder's fixed point theorem, we state and prove the existence results for (1.1). Our view point sheds some new light on problems with weak force potentials and prove that in some situations weak singularities may stimulate the existence of positive solutions.

2 Preliminaries

Let us fix some notation to be used in the following: Given $a \in L^1(0, 1)$, we write $a \succ 0$ if $a \geq 0$ for a.e. $t \in [0, 1]$ and it is positive in a set of positive measure.

It is known that the systems (1.1) is equal to integral equations systems

$$\begin{cases} x(t) = \int_0^1 G(t, s)\big(f_1\big(s, y(s)\big) + e_1(s)\big)ds, \\ y(t) = \int_0^1 G(t, s)\big(f_2\big(s, x(s)\big) + e_2(s)\big)ds, \end{cases}$$

where the Green's function is

$$G(t, s) = \begin{cases} t(1 - s), & 0 \leq t \leq s \leq 1; \\ s(1 - t), & 0 \leq s \leq t \leq 1, \end{cases}$$

and $G(t, s)$ is satisfied the property:

Lemma 2.1 $G(t, s) : [0, 1] \times [0, 1] \to [0, +\infty)$ *is continuous and* $t(1 - t)s(1 - s) \leq G(t, s) \leq t(1 - t)$, *for all* $t \in [0, 1]$, $s \in [0, 1]$.

We define the function $\gamma_i : [0, 1] \to R$ by

$$\gamma_i(t) = \int_0^1 G(t, s)e_i(s)ds, \quad i = 1, 2,$$

which is the unique solution of

$$\begin{cases} -w'' = e_i(t), \\ w(0) = w(1) = 0. \end{cases}$$

Here

$$|\gamma_i(t)| \leq t(1-t)\int_0^1 |e_i(s)|ds.$$

Let us fix some notation to be used in the following: Given function $\gamma \in L^1[0, 1]$, we denote the supremum and infimum by

$$\gamma_i^* = \sup_{t\in(0,1)} \frac{\gamma_i(t)}{t(1-t)} \quad \text{and} \quad \gamma_{i*} = \inf_{t\in(0,1)} \frac{\gamma_i(t)}{t(1-t)},$$

there

$$-\infty < \inf_{t\in(0,1)} \frac{\gamma_i(t)}{t(1-t)} \leq \sup_{t\in(0,1)} \frac{\gamma_i(t)}{t(1-t)} < \infty.$$

3 The case $\gamma_{1*} \geq 0$, $\gamma_{2*} \geq 0$

Theorem 3.1 *Assume that there exists $b_i \succ 0$, $\hat{b}_i \succ 0$ and $0 < \alpha_i < 1$ such that*

(H_1)

$$0 \leq \frac{\hat{b}_i(t)}{x^{\alpha_i}} \leq f_i(t, x) \leq \frac{b_i(t)}{x^{\alpha_i}}, \quad \textit{for all } x > 0, \textit{a.e. } t \in (0, 1), i = 1, 2.$$

(H_2)

$$\int_0^1 b_i(s)\big[s(1-s)\big]^{-\alpha_i} ds < \infty, \quad i = 1, 2.$$

If $\gamma_{1*} \geq 0$, $\gamma_{2*} \geq 0$, then there exists a positive solution of (1.1).

Proof A solution of (1.1) is just a fixed point of the completely continuous map $A(x, y) = (A_1x, A_2y) : C[0, 1] \times C[0, 1] \to C[0, 1] \times C[0, 1]$ defined as

$$\begin{aligned} (A_1x)(t) &:= \int_0^1 G(t,s)\big[f_1\big(s, y(s)\big) + e_1(s)\big]ds \\ &= \int_0^1 G(t,s) f_1\big(s, y(s)\big)ds + \gamma_1(t); \\ (A_2y)(t) &:= \int_0^1 G(t,s)\big[f_2\big(s, x(s)\big) + e_2(s)\big]ds \\ &= \int_0^1 G(t,s) f_2\big(s, x(s)\big)ds + \gamma_2(t). \end{aligned}$$

By a direct application of Schauder's fixed point theorem, the proof is finished if we prove that A maps the closed convex set defined as

$$K = \big\{(x, y) \in C[0,1] \times C[0,1] : r_1 t(1-t) \le x(t) \le R_1 t(1-t), \\ r_2 t(1-t) \le y(t) \le R_2 t(1-t)\big\},$$

and $A : K \to K$ is continuous and compact, where $R_1 > r_1 > 0$, $R_2 > r_2 > 0$ are positive constants to be fixed properly. For convenience, we introduce the following notations

$$\beta_i(t) = \int_0^1 G(t,s) \frac{b_i(s)}{[s(1-s)]^{\alpha_i}} ds,$$

$$\hat{\beta}_i(t) = \int_0^1 G(t,s) \frac{\hat{b}_i(s)}{[s(1-s)]^{\alpha_i}} ds, \quad i = 1, 2,$$

$$\hat{\beta}_{i*} = \inf_{t \in (0,1)} \frac{\hat{\beta}_i(t)}{t(1-t)}, \qquad \hat{\beta}_i^* = \sup_{t \in (0,1)} \frac{\hat{\beta}_i(t)}{t(1-t)},$$

$$\beta_{i*} = \inf_{t \in (0,1)} \frac{\beta_i(t)}{t(1-t)}, \qquad \beta_i^* = \sup_{t \in (0,1)} \frac{\beta_i(t)}{t(1-t)}.$$

Given $(x, y) \in K$, by the nonnegative sign of G and f_i, $i = 1, 2$, we have

$$\begin{aligned}(A_1 x)(t) &= \int_0^1 G(t,s) f_1\big(s, y(s)\big) ds + \gamma_1(t) \\ &\ge \int_0^1 G(t,s) \frac{\hat{b}_1(s)}{y^{\alpha_1}(s)} ds \\ &\ge \int_0^1 G(t,s) \frac{\hat{b}_1(s)}{R_2^{\alpha_1} [s(1-s)]^{\alpha_1}} ds \\ &= \hat{\beta}_1(t) \frac{1}{R_2^{\alpha_1}} \\ &\ge \hat{\beta}_{1*} \cdot \frac{1}{R_2^{\alpha_1}} \big[t(1-t)\big],\end{aligned}$$

and note for every $(x, y) \in K$

$$\begin{aligned}(A_1 x)(t) &= \int_0^1 G(t,s) f_1\big(s, y(s)\big) ds + \gamma_1(t) \\ &\le \int_0^1 G(t,s) \frac{b_1(s)}{y^{\alpha_1}(s)} ds + \gamma_1^* \big[t(1-t)\big] \\ &\le \int_0^1 G(t,s) \frac{b_1(s)}{[r_2 s(1-s)]^{\alpha_1}} ds + \gamma_1^* \big[t(1-t)\big]\end{aligned}$$

$$= \frac{1}{r_2^{\alpha_1}} \int_0^1 G(t,s) \frac{b_1(s)}{[s(1-s)]^{\alpha_1}} ds + \gamma_1^* \big[t(1-t)\big]$$
$$\leq \frac{1}{r_2^{\alpha_1}} t(1-t)\beta_1^* + \gamma_1^* \big[t(1-t)\big]$$
$$= \left[\frac{1}{r_2^{\alpha_1}} \beta_1^* + \gamma_1^*\right]\big[t(1-t)\big].$$

Also, following the same strategy, we have

$$\begin{aligned}
(A_2 y)(t) &= \int_0^1 G(t,s) f_2\big(s, x(s)\big) ds + \gamma_2(t) \\
&\geq \int_0^1 G(t,s) \frac{\hat{b}_2(s)}{x^{\alpha_2}(s)} ds \\
&\geq \int_0^1 G(t,s) \frac{\hat{b}_2(s)}{[R_1 s(1-s)]^{\alpha_2}} ds \\
&= \hat{\beta}_2(t) \cdot \frac{1}{R_1^{\alpha_2}} \\
&\geq \hat{\beta}_{2*} \cdot \frac{1}{R_1^{\alpha_2}} \big[t(1-t)\big], \\
(A_2 y)(t) &= \int_0^1 G(t,s) f_2\big(s, x(s)\big) ds + \gamma_2(t) \\
&\leq \int_0^1 G(t,s) \frac{b_2(s)}{x^{\alpha_2}(s)} ds + \gamma_2^* \big[t(1-t)\big] \\
&\leq \int_0^1 G(t,s) \frac{b_2(s)}{[r_1 s(1-s)]^{\alpha_2}} ds + \gamma_2^* \big[t(1-t)\big] \\
&= \beta_2(t) \frac{1}{r_1^{\alpha_2}} + \gamma_2^* \big[t(1-t)\big] \\
&\leq \left[\beta_2^* \cdot \frac{1}{r_1^{\alpha_2}} + \gamma_2^*\right] \big[t(1-t)\big].
\end{aligned}$$

Thus $(A_1 x, A_2 y) \in K$ if r_1, r_2, R_1 and R_2 are chosen so that

$$\hat{\beta}_{1*} \cdot \frac{1}{R_2^{\alpha_1}} \geq r_1, \qquad \beta_1^* \cdot \frac{1}{r_2^{\alpha_1}} + \gamma_1^* \leq R_1,$$
$$\hat{\beta}_{2*} \cdot \frac{1}{R_1^{\alpha_2}} \geq r_2, \qquad \beta_2^* \cdot \frac{1}{r_1^{\alpha_2}} + \gamma_2^* \leq R_2.$$

Note that $\hat{\beta}_{i*}, \beta_{i*} > 0$ and taking $R = R_1 = R_2$, $r = r_1 = r_2$, $r = \frac{1}{R}$, it is sufficient to find $R > 1$ such that

$$\hat{\beta}_{1*} \cdot R^{1-\alpha_1} \geq 1, \qquad \beta_1^* \cdot R^{\alpha_1} + \gamma_1^* \leq R,$$
$$\hat{\beta}_{2*} \cdot R^{1-\alpha_2} \geq 1, \qquad \beta_2^* \cdot R^{\alpha_2} + \gamma_2^* \leq R,$$

and these inequalities hold for R big enough because $\alpha_i < 1$.

Next we will show that $A : K \to K$ is continuous and compact. Let $x_n, x_0 \in K$ with

$$\|x_n - x_0\| \to 0, \qquad \|y_n - y_0\| \to 0$$

as $n \to \infty$. Here $\|\cdot\|$ is the norm of $C[0,1]$. Also notice that

$$\rho_{1n} = \left| f_1\big(t, y_n(t)\big) - f_1\big(t, y_0(t)\big)\right| \to 0,$$
$$\rho_{2n} = \left| f_2\big(t, x_n(t)\big) - f_2\big(t, x_0(t)\big)\right| \to 0,$$

as $n \to \infty$, $t \in (0,1)$, and

$$\rho_{1n} \leq f_1\big(t, y_n(t)\big) + f_1\big(t, y_0(t)\big), \quad t \in (0,1);$$
$$\rho_{2n} \leq f_2\big(t, x_n(t)\big) + f_2\big(t, x_0(t)\big), \quad t \in (0,1).$$

Here

$$f_1\big(t, y_n(t)\big) \leq \frac{b_1(t)}{y_n^{\alpha_1}(t)} \leq \frac{b_1(t)}{r_2^{\alpha_1}[t(1-t)]^{\alpha_1}}, \quad t \in (0,1);$$
$$f_1\big(t, y_0(t)\big) \leq \frac{b_1(t)}{y_0^{\alpha_1}(t)} \leq \frac{b_1(t)}{r_2^{\alpha_1}[t(1-t)]^{\alpha_1}}, \quad t \in (0,1);$$
$$f_2\big(t, x_n(t)\big) \leq \frac{b_2(t)}{x_n^{\alpha_2}(t)} \leq \frac{b_2(t)}{r_1^{\alpha_2}[t(1-t)]^{\alpha_2}}, \quad t \in (0,1);$$
$$f_2\big(t, x_0(t)\big) \leq \frac{b_2(t)}{x_0^{\alpha_2}(t)} \leq \frac{b_2(t)}{r_1^{\alpha_2}[t(1-t)]^{\alpha_2}}, \quad t \in (0,1).$$

These together with the Lebesgue dominated convergence theorem guarantee that

$$\|A_1 x_n - A_1 x_0\| \leq \sup_{t\in[0,1]} \int_0^1 G(t,s)\rho_{1n}(s)ds \to 0,$$
$$\|A_2 y_n - A_2 y_0\| \leq \sup_{t\in[0,1]} \int_0^1 G(t,s)\rho_{2n}(s)ds \to 0,$$

as $n \to \infty$. So, $A : K \to K$ is continuous.

We already know that $A(K) \subset K$ and obviously K is bounded, so $A(K)$ is bounded.

Let

$$\delta=\min\left\{\frac{\epsilon}{r_2^{-\alpha_1}\int_0^1 b_1(s)[s(1-s)]^{-\alpha_1}ds+\int_0^1|e_1(s)|ds},\right.$$
$$\left.\frac{\epsilon}{r_1^{-\alpha_2}\int_0^1 b_2(s)[s(1-s)]^{-\alpha_2}ds+\int_0^1|e_2(s)|ds}\right\},$$

then for any $\epsilon>0$, $t,t'\in[0,1]$, $|t-t'|<\delta$, we have

$$\left|G(t,s)-G\left(t',s\right)\right|\le\left|t-t'\right|<\frac{\epsilon}{r_2^{-\alpha_1}\int_0^1 b_1(s)[s(1-s)]^{-\alpha_1}ds+\int_0^1|e_1(s)|ds},$$
$$\left|G(t,s)-G\left(t',s\right)\right|\le\left|t-t'\right|<\frac{\epsilon}{r_1^{-\alpha_2}\int_0^1 b_2(s)[s(1-s)]^{-\alpha_2}ds+\int_0^1|e_2(s)|ds}.$$

Thus we have

$$\begin{aligned}
&\left|(A_1x)(t)-(A_1x)\left(t'\right)\right|\\
&\quad=\left|\int_0^1\left[G(t,s)-G\left(t',s\right)\right]\left[f_1\left(s,y(s)\right)+e_1(s)\right]ds\right|\\
&\quad\le\int_0^1\left|\left[G(t,s)-G\left(t',s\right)\right]\right|\left|\left[f_1\left(s,y(s)\right)+e_1(s)\right]\right|ds\\
&\quad\le\int_0^1\left|\left[G(t,s)-G\left(t',s\right)\right]\right|\left[b_1(s)\frac{1}{r_2^{\alpha_1}[s(1-s)]^{\alpha_1}}+\left|e_1(s)\right|\right]ds\\
&\quad<\epsilon,\\
&\left|(A_2y)(t)-(A_2y)\left(t'\right)\right|\\
&\quad=\left|\int_0^1\left[G(t,s)-G\left(t',s\right)\right]\left[f_2\left(s,x(s)\right)+e_2(s)\right]ds\right|\\
&\quad\le\int_0^1\left|\left[G(t,s)-G\left(t',s\right)\right]\right|\left|\left[f_2\left(s,x(s)\right)+e_2(s)\right]\right|ds\\
&\quad\le\int_0^1\left|\left[G(t,s)-G\left(t',s\right)\right]\right|\left[b_2(s)\frac{1}{r_1^{\alpha_2}[s(1-s)]^{\alpha_2}}+\left|e_2(s)\right|\right]ds\\
&\quad<\epsilon.
\end{aligned}$$

Then the Arzela–Ascoli theorem guarantees that $A:K\to K$ is compact. □

4 The case $\gamma_1^*\le 0$, $\gamma_2^*\le 0$

The aim of this section is to show that the presence of a weak nonlinearity makes it possible to find positive solutions if $\gamma_1^*\le 0$, $\gamma_2^*\le 0$.

Theorem 4.1 *There exist* $b_i, \hat{b}_i \succ 0$ *and* $0 < \alpha_i < 1$, *such that* (H_1) *and* (H_2) *are satisfied. If* $\gamma_1^* \leq 0$, $\gamma_2^* \leq 0$, *and*

$$\begin{aligned} \gamma_{1*} &\geq \left[\alpha_1\alpha_2 \cdot \frac{\hat{\beta}_{1*}}{(\beta_2^*)^{\alpha_1}}\right]^{\frac{1}{1-\alpha_1\alpha_2}} \left(1 - \frac{1}{\alpha_1\alpha_2}\right), \\ \gamma_{2*} &\geq \left[\alpha_1\alpha_2 \cdot \frac{\hat{\beta}_{2*}}{(\beta_1^*)^{\alpha_2}}\right]^{\frac{1}{1-\alpha_1\alpha_2}} \left(1 - \frac{1}{\alpha_1\alpha_2}\right), \end{aligned} \tag{4.1}$$

then there exists a positive solution of (1.1).

Proof In this case, to prove that $A : K \to K$, it is sufficient to find $0 < r_1 < R_1$, $0 < r_2 < R_2$ such that

$$\frac{\hat{\beta}_{1*}}{R_2^{\alpha_1}} + \gamma_{1*} \geq r_1, \qquad \frac{\beta_1^*}{r_2^{\alpha_1}} \leq R_1, \tag{4.2}$$

$$\frac{\hat{\beta}_{2*}}{R_1^{\alpha_2}} + \gamma_{2*} \geq r_2, \qquad \frac{\beta_2^*}{r_1^{\alpha_2}} \leq R_2. \tag{4.3}$$

If we fix $R_1 = \frac{\beta_1^*}{r_2^{\alpha_1}}$, $R_2 = \frac{\beta_2^*}{r_1^{\alpha_2}}$, then the first inequality of (4.3) holds if r_2 satisfies

$$\hat{\beta}_{2*}\left(\beta_1^*\right)^{-\alpha_2} r_2^{\alpha_1\alpha_2} + \gamma_{2*} \geq r_2,$$

or equivalently

$$\gamma_{2*} \geq g(r_2) := r_2 - \frac{\hat{\beta}_{2*}}{(\beta_1^*)^{\alpha_2}} r_2^{\alpha_1\alpha_2}.$$

The function $g(r_2)$ possesses a minimum at

$$r_{20} := \left[\alpha_1\alpha_2 \cdot \frac{\hat{\beta}_{2*}}{(\beta_1^*)^{\alpha_2}}\right]^{\frac{1}{1-\alpha_1\alpha_2}}.$$

Taking $r_2 = r_{20}$, then (4.3) holds if

$$\gamma_{2*} \geq g(r_{20}) = \left[\alpha_1\alpha_2 \cdot \frac{\hat{\beta}_{2*}}{(\beta_1^*)^{\alpha_2}}\right]^{\frac{1}{1-\alpha_1\alpha_2}} \left(1 - \frac{1}{\alpha_1\alpha_2}\right).$$

Similarly,

$$\gamma_{1*} \geq h(r_1) := r_1 - \frac{\hat{\beta}_{1*}}{(\beta_2^*)^{\alpha_1}} r_1^{\alpha_1\alpha_2},$$

$h(r_1)$ possesses a minimum at

$$r_{10} := \left[\alpha_1\alpha_2 \cdot \frac{\hat{\beta}_{1*}}{(\beta_2^*)^{\alpha_1}}\right]^{\frac{1}{1-\alpha_1\alpha_2}},$$

$$\gamma_{1*} \geq \left[\alpha_1\alpha_2 \cdot \frac{\hat{\beta}_{1*}}{(\beta_2^*)^{\alpha_1}}\right]^{\frac{1}{1-\alpha_1\alpha_2}} \left(1 - \frac{1}{\alpha_1\alpha_2}\right).$$

Taking $r_1 = r_{10}, r_2 = r_{20}$, then the first inequalities in (4.2) and (4.3) hold if $\gamma_{1*} \geq g(r_1)$ and $\gamma_{2*} \geq g(r_2)$, which are just condition (4.2). The second inequalities hold directly from the choice of R_1 and R_2, so it remains to prove that $R_1 = \frac{\beta_1^*}{r_{20}^{\alpha_1}} > r_{10}$, $R_2 = \frac{\beta_2^*}{r_{10}^{\alpha_2}} > r_{20}$. This is easily verified through elementary computations:

$$\begin{aligned} R_1 &= \frac{\beta_1^*}{r_{20}^{\alpha_1}} = \frac{\beta_1^*}{\{[\alpha_1\alpha_2 \cdot \frac{\hat{\beta}_{2*}}{(\beta_1^*)^{\alpha_2}}]^{\frac{1}{1-\alpha_1\alpha_2}}\}^{\alpha_1}} \\ &= \frac{\beta_1^*}{[\alpha_1\alpha_2 \cdot \frac{\hat{\beta}_{2*}}{(\beta_1^*)^{\alpha_2}}]^{\frac{\alpha_1}{1-\alpha_1\alpha_2}}} = \frac{(\beta_1^*)^{1+\frac{\alpha_1\alpha_2}{1-\alpha_1\alpha_2}}}{(\alpha_1\alpha_2 \cdot \hat{\beta}_{2*})^{\frac{\alpha_1}{1-\alpha_1\alpha_2}}} \\ &= \frac{(\beta_1^*)^{\frac{1}{1-\alpha_1\alpha_2}}}{[(\alpha_1\alpha_2 \cdot \hat{\beta}_{2*})^{\alpha_1}]^{\frac{1}{1-\alpha_1\alpha_2}}} = \left[\frac{\beta_1^*}{(\alpha_1\alpha_2 \cdot \hat{\beta}_{2*})^{\alpha_1}}\right]^{\frac{1}{1-\alpha_1\alpha_2}} \\ &= \left[\frac{1}{(\alpha_1\alpha_2)^{\alpha_1}} \cdot \frac{\beta_1^*}{(\hat{\beta}_{2*})^{\alpha_1}}\right]^{\frac{1}{1-\alpha_1\alpha_2}} > \left[\alpha_1\alpha_2 \cdot \frac{\hat{\beta}_{1*}}{(\beta_2^*)^{\alpha_1}}\right]^{\frac{1}{1-\alpha_1\alpha_2}} = r_{10}, \end{aligned}$$

since $\hat{\beta}_{i*} \leq \beta_i^*$, $i = 1, 2$. Similarly, we have $R_2 > r_{20}$. □

5 The case $\gamma_{1*} \geq 0$, $\gamma_2^* \leq 0$ ($\gamma_1^* \leq 0$, $\gamma_{2*} \geq 0$)

Theorem 5.1 *Assume* (H$_1$) *and* (H$_2$) *are satisfied. If* $\gamma_{1*} \geq 0$, $\gamma_2^* \leq 0$ *and*

$$\gamma_{2*} \geq r_{21} - \hat{\beta}_{2*} \cdot \frac{r_{21}^{\alpha_1\alpha_2}}{(\beta_1^* + \gamma_1^* r_{21}^{\alpha_1})^{\alpha_2}}, \tag{5.1}$$

where $0 < r_{21} < +\infty$ *is a unique positive solution of the equation*

$$r_2^{1-\alpha_1\alpha_2}\left(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\right)^{1+\alpha_2} = \alpha_1\alpha_2\beta_1^*\hat{\beta}_{2*}, \tag{5.2}$$

then there exists a positive solution of (1.1).

Proof We follow the same strategy and notation as in the proof of ahead theorem. In this case, to prove that $A : K \to K$, it is sufficient to find $r_1 < R_1, r_2 < R_2$ such that

$$\frac{\hat{\beta}_{1*}}{R_2^{\alpha_1}} \geq r_1, \qquad \frac{\beta_2^*}{r_1^{\alpha_2}} \leq R_2. \tag{5.3}$$

$$\frac{\hat{\beta}_{2*}}{R_1^{\alpha_2}} + \gamma_{2*} \geq r_2, \qquad \frac{\beta_1^*}{r_2^{\alpha_1}} + \gamma_1^* \leq R_1. \tag{5.4}$$

If we fix $R_2 = \frac{\beta_2^*}{r_1^{\alpha_2}}$, then the first inequality of (5.3) holds if r_1 satisfies

$$\frac{\hat{\beta}_{1*}}{(\beta_2^*)^{\alpha_1}} \cdot r_1^{\alpha_1\alpha_2} \geq r_1, \tag{5.5}$$

or equivalently

$$0 < r_1 \leq \left[\frac{\hat{\beta}_{1*}}{(\beta_2^*)^{\alpha_1}}\right]^{\frac{1}{1-\alpha_1\alpha_2}}. \tag{5.6}$$

If we chose $r_1 > 0$ small enough, then (5.6) holds, and R_2 is big enough.

If we fix $R_1 = \frac{\beta_1^*}{r_2^{\alpha_1}} + \gamma_1^*$ then the first inequality of (5.4) holds if r_2 satisfies

$$\begin{aligned}
\gamma_{2*} &\geq r_2 - \frac{\hat{\beta}_{2*}}{R_1^{\alpha_2}} \\
&= r_2 - \hat{\beta}_{2*} \cdot \frac{1}{(\frac{\beta_1^*}{r_2^{\alpha_1}} + \gamma_1^*)^{\alpha_2}} \\
&= r_2 - \hat{\beta}_{2*} \cdot \frac{1}{(\frac{\beta_1^*+\gamma_1^*\cdot r_2^{\alpha_1}}{r_2^{\alpha_1}})^{\alpha_2}} \\
&= r_2 - \hat{\beta}_{2*} \cdot \frac{r_2^{\alpha_1\alpha_2}}{(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{\alpha_2}},
\end{aligned}$$

or equivalently

$$\gamma_{2*} \geq f(r_2) := r_2 - \hat{\beta}_{2*} \cdot \frac{r_2^{\alpha_1\alpha_2}}{(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{\alpha_2}}. \tag{5.7}$$

According to

$$\begin{aligned}
f'(r_2) &= 1 - \hat{\beta}_{2*} \cdot \frac{1}{(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{2\alpha_2}} \cdot \big[\alpha_1\alpha_2 r_2^{\alpha_1\alpha_2-1}\big(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\big)^{\alpha_2} \\
&\quad - r_2^{\alpha_1\alpha_2}\alpha_2\big(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\big)^{\alpha_2-1}\alpha_1\gamma_1^* r_2^{\alpha_1-1}\big] \\
&= 1 - \frac{\hat{\beta}_{2*}\alpha_1\alpha_2 r_2^{\alpha_1\alpha_2-1}}{(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{\alpha_2}}\left[1 - \frac{r_2^{\alpha_1}\gamma_1^*}{\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}}\right] \\
&= 1 - \alpha_1\alpha_2\beta_1^*\hat{\beta}_{2*} r_2^{\alpha_1\alpha_2-1}\big(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\big)^{-1-\alpha_2},
\end{aligned} \tag{5.8}$$

we have $f'(0) = -\infty$, $f'(+\infty) = 1$, then there exists r_{21} such that $f'(r_{21}) = 0$, and

$$\begin{aligned}
f''(r_2) &= -\big[\alpha_1\alpha_2\beta_1^*\hat{\beta}_{2*}(\alpha_1\alpha_2 - 1)r_2^{\alpha_1\alpha_2-2}\big(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\big)^{-1-\alpha_2} \\
&\quad + \alpha_1\alpha_2\beta_1^*\hat{\beta}_{2*} r_2^{\alpha_1\alpha_2-1}(-1-\alpha_2)\big(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\big)^{-2-\alpha_2}\gamma_1^*\alpha_1 r_2^{\alpha_1-1}\big] \\
&> 0.
\end{aligned} \tag{5.9}$$

Then the function $f(r_2)$ possesses a minimum at r_{21}, i.e., $f(r_{21}) = \min_{r_2 \in (0,+\infty)} f(r_2)$.
Note $f'(r_{21}) = 0$, then we have

$$1 - \alpha_1\alpha_2\beta_1^*\hat{\beta}_{2*}r_{21}^{\alpha_1\alpha_2-1}\left(\beta_1^* + \gamma_1^* \cdot r_{21}^{\alpha_1}\right)^{-1-\alpha_2} = 0,$$

or equivalently,

$$r_{21}^{1-\alpha_1\alpha_2}\left(\beta_1^* + \gamma_1^* \cdot r_{21}^{\alpha_1}\right)^{1+\alpha_2} = \alpha_1\alpha_2\beta_1^*\hat{\beta}_{2*}. \tag{5.10}$$

Taking $r_2 = r_{21}$, then the first inequality in (5.4) holds if $\gamma_{2*} \geq f(r_{21})$, which is just condition (5.1). The second inequality holds directly by the choice of R_2, and it would remain to prove that $r_{21} < R_2$ and $r_{10} < R_1$. These inequalities hold for R_2 big enough and r_1 small enough. □

Similarly, we have the following Theorem.

Theorem 5.2 *Assume* (H$_1$) *and* (H$_2$) *are satisfied. If* $\gamma_1^* \leq 0$, $\gamma_{2*} \geq 0$ *and*

$$\gamma_{1*} \geq r_{11} - \hat{\beta}_{1*} \cdot \frac{r_{11}^{\alpha_1\alpha_2}}{(\beta_2^* + \gamma_2^* r_{11}^{\alpha_2})^{\alpha_1}}, \tag{5.11}$$

where $0 < r_{11} < +\infty$ *is a unique positive solution of the equation*

$$r_1^{1-\alpha_1\alpha_2}\left(\beta_2^* + \gamma_2^* \cdot r_1^{\alpha_2}\right)^{1+\alpha_1} = \alpha_1\alpha_2\beta_2^*\hat{\beta}_{1*},$$

then there exists a positive solution of (1.1).

6 The case $\gamma_{1*} < 0 < \gamma_1^*$, $\gamma_{2*} < 0 < \gamma_2^*$

Theorem 6.1 *Assume* (H$_1$) *and* (H$_2$) *are satisfied. If* $\gamma_{1*} < 0 < \gamma_1^*$, $\gamma_{2*} < 0 < \gamma_2^*$ *and*

$$\gamma_{1*} \geq r_{10} - \hat{\beta}_{1*} \cdot \frac{r_{10}^{\alpha_1\alpha_2}}{(\beta_2^* + \gamma_2^* r_{10}^{\alpha_2})^{\alpha_1}}, \tag{6.1}$$

$$\gamma_{2*} \geq r_{20} - \hat{\beta}_{2*} \cdot \frac{r_{20}^{\alpha_1\alpha_2}}{(\beta_1^* + \gamma_1^* r_{20}^{\alpha_1})^{\alpha_2}}, \tag{6.2}$$

where $0 < r_{10} < +\infty$ *is a unique positive solution of the equation*

$$r_1^{1-\alpha_1\alpha_2}\left(\beta_2^* + \gamma_2^* \cdot r_1^{\alpha_2}\right)^{1+\alpha_1} = \alpha_1\alpha_2\beta_2^*\hat{\beta}_{1*}, \tag{6.3}$$

and $0 < r_{20} < +\infty$ *is a unique positive solution of the equation*

$$r_2^{1-\alpha_1\alpha_2}\left(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\right)^{1+\alpha_2} = \alpha_1\alpha_2\beta_1^*\hat{\beta}_{2*}, \tag{6.4}$$

then there exists a positive solution of (1.1).

Proof We follow the same strategy and notation as in the proof of ahead theorem. In this case, to prove that $A : K \to K$, it is sufficient to find $r_1 < R_1, r_2 < R_2$ such that

$$\frac{\hat{\beta}_{1*}}{R_2^{\alpha_1}} + \gamma_{1*} \geq r_1, \qquad \frac{\beta_1^*}{r_2^{\alpha_1}} + \gamma_1^* \leq R_1. \tag{6.5}$$

$$\frac{\hat{\beta}_{2*}}{R_1^{\alpha_2}} + \gamma_{2*} \geq r_2, \qquad \frac{\beta_2^*}{r_1^{\alpha_2}} + \gamma_2^* \leq R_2. \tag{6.6}$$

If we fix $R_1 = \frac{\beta_1^*}{r_2^{\alpha_1}} + \gamma_1^*$, $R_2 = \frac{\beta_2^*}{r_1^{\alpha_2}} + \gamma_2^*$, then the first inequality of (6.6) holds if r_2 satisfies

$$\gamma_{2*} \geq g(r_2) := r_2 - \hat{\beta}_{2*} \cdot \frac{r_2^{\alpha_1\alpha_2}}{(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{\alpha_2}}. \tag{6.7}$$

Following the same calculation as in the proof of (5.7),

$$g'(r_2) = 1 - \alpha_1\alpha_2\beta_1^*\hat{\beta}_{2*}r_2^{\alpha_1\alpha_2-1}\left(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\right)^{-1-\alpha_2}, \tag{6.8}$$

we have $g'(0) = -\infty$, $g'(+\infty) = 1$, then there exists r_{20} such that $g'(r_{20}) = 0$, and

$$\begin{aligned} g''(r_2) = &-\Big[\alpha_1\alpha_2\beta_1^*\hat{\beta}_{2*}(\alpha_1\alpha_2 - 1)r_2^{\alpha_1\alpha_2-2}\left(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\right)^{-1-\alpha_2} \\ &+ \alpha_1\alpha_2\beta_1^*\hat{\beta}_{2*}r_2^{\alpha_1\alpha_2-1}(-1-\alpha_2)\left(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\right)^{-2-\alpha_2}\gamma_1^*\alpha_1 r_2^{\alpha_1-1}\Big] \\ &> 0. \end{aligned} \tag{6.9}$$

Then the function $g(r_2)$ possesses a minimum at r_{20}, i.e., $g(r_{20}) = \min_{r_2 \in (0,+\infty)} g(r_2)$.

Note $g'(r_{20}) = 0$, then we have

$$r_{20}^{1-\alpha_1\alpha_2}\left(\beta_1^* + \gamma_1^* \cdot r_{20}^{\alpha_1}\right)^{1+\alpha_2} = \alpha_1\alpha_2\beta_1^*\hat{\beta}_{2*}. \tag{6.10}$$

Similarly,

$$\gamma_{1*} \geq g(r_1) := r_1 - \hat{\beta}_{1*} \cdot \frac{r_1^{\alpha_1\alpha_2}}{(\beta_2^* + \gamma_2^* \cdot r_1^{\alpha_2})^{\alpha_1}}. \tag{6.11}$$

$g(r_{10}) = \min_{r_1 \in (0,+\infty)} g(r_1)$, and

$$r_{10}^{1-\alpha_1\alpha_2}\left(\beta_2^* + \gamma_2^* \cdot r_{10}^{\alpha_2}\right)^{1+\alpha_1} = \alpha_1\alpha_2\beta_2^*\hat{\beta}_{1*}. \tag{6.12}$$

Taking $r_1 = r_{10}$ and $r_2 = r_{20}$, then the first inequality in (6.5) and (6.6) hold if $\gamma_{1*} \geq g(r_{10})$, $\gamma_{2*} \geq g(r_{20})$, which are just condition (6.1) and (6.2). The second inequalities hold directly by the choice of R_1 and R_2, and it would remain to prove that $r_{10} < R_1$ and $r_{20} < R_2$. This is easily verified through elementary computations.

$$R_1 = \frac{\beta_1^*}{r_{20}^{\alpha_1}} + \gamma_1^*$$

$$= \frac{\beta_1^* + \gamma_1^* \cdot r_{20}^{\alpha_1}}{r_{20}^{\alpha_1}}$$

$$= \frac{(\alpha_1\alpha_2\beta_1^*\hat{\beta}_{2*})^{\frac{1}{1+\alpha_2}} \cdot r_{20}^{\frac{\alpha_1\alpha_2-1}{1+\alpha_2}}}{r_{20}^{\alpha_1}}$$

$$= \left(\alpha_1\alpha_2\beta_1^*\hat{\beta}_{2*}\right)^{\frac{1}{1+\alpha_2}} \cdot r_{20}^{-\frac{1+\alpha_1}{1+\alpha_2}}.$$

The proof is the same as that in R_1, $R_2 = (\alpha_1\alpha_2\beta_2^*\hat{\beta}_{1*})^{\frac{1}{1+\alpha_1}} \cdot r_{10}^{-\frac{1+\alpha_2}{1+\alpha_1}}$.

Next, we will prove $r_{10} < R_1, r_{20} < R_2$, or equivalently,

$$\begin{aligned} r_{10}r_{20}^{\frac{1+\alpha_1}{1+\alpha_2}} &< \left(\alpha_1\alpha_2\beta_1^*\hat{\beta}_{2*}\right)^{\frac{1}{1+\alpha_2}}, \\ r_{20}r_{10}^{\frac{1+\alpha_2}{1+\alpha_1}} &< \left(\alpha_1\alpha_2\beta_2^*\hat{\beta}_{1*}\right)^{\frac{1}{1+\alpha_1}}. \end{aligned} \tag{6.13}$$

Namely,

$$r_{10}^{1+\alpha_2}r_{20}^{1+\alpha_1} < \alpha_1\alpha_2\beta_1^*\hat{\beta}_{2*}, r_{20}^{1+\alpha_1}r_{10}^{1+\alpha_2} < \alpha_1\alpha_2\beta_2^*\hat{\beta}_{1*}. \tag{6.14}$$

On the other hand,

$$r_{20}^{1-\alpha_1\alpha_2}\left(\beta_1^*\right)^{1+\alpha_2} \le \alpha_1\alpha_2\beta_1^*\hat{\beta}_{2*}.$$

Then

$$r_{20} \le \left(\alpha_1\alpha_2\left(\beta_1^*\right)^{-\alpha_2}\hat{\beta}_{2*}\right)^{\frac{1}{1-\alpha_1\alpha_2}}. \tag{6.15}$$

Similarly

$$r_{10} \le \left(\alpha_1\alpha_2\left(\beta_2^*\right)^{-\alpha_1}\hat{\beta}_{1*}\right)^{\frac{1}{1-\alpha_1\alpha_2}}. \tag{6.16}$$

By (6.15) and (6.16),

$$r_{10}^{1+\alpha_2}r_{20}^{1+\alpha_1} \le \left(\alpha_1\alpha_2\left(\beta_2^*\right)^{-\alpha_1}\hat{\beta}_{1*}\right)^{\frac{1+\alpha_2}{1-\alpha_1\alpha_2}}\left(\alpha_1\alpha_2\left(\beta_1^*\right)^{-\alpha_2}\hat{\beta}_{2*}\right)^{\frac{1+\alpha_1}{1-\alpha_1\alpha_2}}.$$

Now if we can prove

$$\left(\alpha_1\alpha_2\left(\beta_2^*\right)^{-\alpha_1}\hat{\beta}_{1*}\right)^{\frac{1+\alpha_2}{1-\alpha_1\alpha_2}}\left(\alpha_1\alpha_2\left(\beta_1^*\right)^{-\alpha_2}\hat{\beta}_{2*}\right)^{\frac{1+\alpha_1}{1-\alpha_1\alpha_2}} < \alpha_1\alpha_2\beta_1^*\hat{\beta}_{2*}, \tag{6.17}$$

then

$$r_{10}^{1+\alpha_2}r_{20}^{1+\alpha_1} < \alpha_1\alpha_2\beta_1^*\hat{\beta}_{2*}.$$

In fact,

$$(\alpha_1\alpha_2)^{\frac{2+\alpha_2+\alpha_1-1}{1-\alpha_1\alpha_2}} \cdot \left(\frac{\hat{\beta}_{1*}}{\beta_1^*}\right)^{\frac{1+\alpha_2}{1-\alpha_1\alpha_2}} \cdot \left(\frac{\hat{\beta}_{2*}}{\beta_2^*}\right)^{\frac{\alpha_1(1+\alpha_2)}{1-\alpha_1\alpha_2}} < 1,$$

since $\hat{\beta}i* \le \beta_i^*$, $i = 1, 2$. Similarly, we have $r_{20}^{1+\alpha_1} r_{10}^{1+\alpha_2} < \alpha_1\alpha_2\beta_2^*\hat{\beta}_{1*}$, we omit the details. Now we can obtain $r_{10} < R_1, r_{20} < R_2$. The proof is complete. □

As the method of proof is similar, we omitted proof of Theorems 7.1, 7.2, 8.1 and 8.2. We only give the conclusions and theorems.

7 The case $\gamma_1^* \le 0$, $\gamma_{2*} < 0 < \gamma_2^*$ ($\gamma_2^* \le 0$, $\gamma_{1*} < 0 < \gamma_1^*$)

Theorem 7.1 *Assume* (H$_1$) *and* (H$_2$) *are satisfied. If* $\gamma_1^* \le 0$, $\gamma_{2*} < 0 < \gamma_2^*$ *and*

$$\gamma_{2*} \ge \left(1 - \frac{1}{\alpha_1\alpha_2}\right)\left[\alpha_1\alpha_2 \frac{\hat{\beta}_{2*}}{(\beta_1^*)^{\alpha_2}}\right]^{\frac{1}{1-\alpha_1\alpha_2}}, \tag{7.1}$$

$$\gamma_{1*} \ge r_{11} - \hat{\beta}_{1*} \cdot \frac{r_{11}^{\alpha_1\alpha_2}}{(\beta_2^* + \gamma_2^* r_{11}^{\alpha_2})^{\alpha_1}}, \tag{7.2}$$

where $0 < r_{11} < +\infty$ *is a unique positive solution of the equation*

$$r_1^{1-\alpha_1\alpha_2}\left(\beta_2^* + \gamma_2^* \cdot r_1^{\alpha_2}\right)^{1+\alpha_1} = \alpha_1\alpha_2\beta_2^*\hat{\beta}_{1*}, \tag{7.3}$$

then there exists a positive solution of (1.1).

Theorem 7.2 *Assume* (H$_1$) *and* (H$_2$) *are satisfied. If* $\gamma_2^* \le 0$, $\gamma_{1*} < 0 < \gamma_1^*$ *and*

$$\gamma_{1*} \ge \left(1 - \frac{1}{\alpha_1\alpha_2}\right) \cdot \left[\alpha_1\alpha_2 \frac{\hat{\beta}_{1*}}{(\beta_2^*)^{\alpha_1}}\right]^{\frac{1}{1-\alpha_1\alpha_2}}, \tag{7.4}$$

$$\gamma_{2*} \ge r_{21} - \hat{\beta}_{2*} \cdot \frac{r_{21}^{\alpha_1\alpha_2}}{(\beta_1^* + \gamma_1^* r_{21}^{\alpha_1})^{\alpha_2}}, \tag{7.5}$$

where $0 < r_{21} < +\infty$ *is a unique positive solution of the equation*

$$r_2^{1-\alpha_1\alpha_2}\left(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\right)^{1+\alpha_2} = \alpha_1\alpha_2\beta_1^*\hat{\beta}_{2*},$$

then there exists a positive solution of (1.1).

8 The case $\gamma_{1*} \ge 0$, $\gamma_{2*} < 0 < \gamma_2^*$ ($\gamma_{2*} \ge 0$, $\gamma_{1*} < 0 < \gamma_1^*$)

Theorem 8.1 *Assume* (H$_1$) *and* (H$_2$) *are satisfied. If* $\gamma_{1*} \ge 0$, $\gamma_{2*} < 0 < \gamma_2^*$ *and*

$$\gamma_{2*} \ge r_{22} - \hat{\beta}_{2*} \cdot \frac{r_{22}^{\alpha_1\alpha_2}}{(\beta_1^* + \gamma_1^* r_{22}^{\alpha_1})^{\alpha_2}}, \tag{8.1}$$

where $0 < r_{22} < +\infty$ *is a unique positive solution of the equation*

$$r_2^{1-\alpha_1\alpha_2}\left(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}\right)^{1+\alpha_2} = \alpha_1\alpha_2\beta_1^*\hat{\beta}_{2*}, \tag{8.2}$$

then there exists a positive solution of (1.1).

Theorem 8.2 *Assume* (H_1) *and* (H_2) *are satisfied. If* $\gamma_{2*} \geq 0$, $\gamma_{1*} < 0 < \gamma_1^*$ *and*

$$\gamma_{1*} \geq r_{12} - \hat{\beta}_{1*} \cdot \frac{r_{12}^{\alpha_1 \alpha_2}}{(\beta_2^* + \gamma_2^* r_{12}^{\alpha_2})^{\alpha_1}}, \tag{8.3}$$

where $0 < r_{12} < +\infty$ *is a unique positive solution of the equation*

$$r_1^{1-\alpha_1\alpha_2}\left(\beta_2^* + \gamma_2^* \cdot r_1^{\alpha_2}\right)^{1+\alpha_1} = \alpha_1\alpha_2\beta_2^*\hat{\beta}_{1*},$$

then there exists a positive solution of (1.1).

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